Escape from a fluctuating double well

Antonio J. R. Madureira and Peter Hänggi
Department of Physics, University of Augsburg, Memminger Strasse 6, D-86135 Augsburg, Germany

Vincent Buonomano and Waldyr A. Rodrigues Jr.
Departamento de Matematica Aplicada, Universitade Estadual de Campinas, Caixa Postal 6065, 13081-970 Campinas, Brazil

(Received 22 September 1994; revised manuscript received 20 December 1994)

We study thermally driven escape from a double well over a fluctuating barrier height. The fluctuations of the bistable potential are governed by exponentially correlated Gaussian noise of weak-to-moderate-to-large noise correlation time τ. Exact results are obtained for the limiting cases of very fast ($\tau \to 0$) and very slow ($\tau \to \infty$) barrier fluctuations. For finite noise color τ, we present approximation schemes for the stochastic dynamics of nonlinear systems that are driven simultaneously by both a white noise source and a multiplicative colored noise (colored noise driven parametric stochastic flows). Our approximative results for arbitrary, but finite noise color τ become exact for escape in a piecewise parabolic bistable potential with a cusp at the transition state.

PACS number(s): 05.40.+j, 82.20.Mj

I. INTRODUCTION

Ever since the pioneering contribution of Svante Arrhenius and Hendrik Antoine Kramers, the problem of escape from metastable states has become ubiquitous in almost all scientific areas. A particular interesting variety of this problem is the transport in complex systems as it occurs in glasses [1,2] or in biological systems [3,4], possessing many metastable states. Typically these complex systems are *open systems,* being in contact with one or more fluctuating environments. Then the fluctuations are no longer related to dissipation via a fluctuation-dissipation theorem of the Einstein-Nyquist type [5,6] which relates the friction strength to the correlation properties of fluctuations. Clearly, escape from states of local stability can occur—in the absence of quantum tunneling—via noise-assisted hopping events only. For such nonequilibrium systems the problems of evaluating the escape time thus becomes a daunting problem, because even the stationary probability generally is no longer given by the Boltzmann distribution; it must be determined, instead, self-consistently from the fluctuation properties. Within this context, an interesting variation of the common topic of escape from a metastable state [7,8] arises when the barrier configuration is no longer static, but is a fluctuating quantity itself [9–12]. Hence, the topic is closely related to the area of noise-assisted escape in metastable fields in the presence of fluctuating (i.e., nonquenched) control parameters. This latter theme was to the authors best knowledge addressed first in Ref. [12], and pursued further within a different spirit by others in Refs. [13–15].

As emphasized by the advocates of Refs. [9–11], the problem of surmounting fluctuating barriers involves several relevant time scales. Most importantly, because stochastic barrier modulations likely are induced by strong couplings to relevant degrees of freedom of the system, the fluctuations of the potential landscape may vary on a time scale much slower, be comparable, or even may vary on a much larger scale as compared to the typical average time scale for local relaxation. The theme is thus necessarily related in spirit to the area of colored noise driven escape [16]; an area which by itself has attracted a tremendous amount of activity within the last decade [17,18]. The previous studies on escape in fluctuating metastable potentials [9–11,12,13] had been subject to severe limitations: These are either (i) the statistics of the noise sources is approximated by white Gaussian noise [9,10,12,13] or by a two-state noise (dichotomic noise) [11] and/or (ii) the form of the potential has been restricted [11(a)–(d)] to piecewise linear barriers and wells.

The objective here is the study of the stochastic dynamics of a single relevant degree of freedom which is driven simultaneously by white Gaussian noise and colored noise. This situation is generic for a variety of physical situations. A first example is biological transport (see above) which works in the presence of white thermal noise and internal, generally correlated random noise of biological origin, such as, e.g., the hydrolysis mechanism of adenosine 5′-triphosphate [19] (ATP). Another physical situation is given in nonlinear optics such as the dynamics of a dye laser [15,20–23]. In this latter case colored noise originates from the pump fluctuations while the white noise models the effect of quantum fluctuations. In contrast to the case with a single colored noise source [16–18,21,24–29] there exist relatively few prior studies [9–13,21(b),22,30–32] wherein one accounts systematically for the mutual influence of white and colored noise, driving simultaneously a single degree of freedom. Indeed, none of the prior approximate schemes correctly describes a linear dynamics driven simultaneously by additive white and additive colored fluctuations of arbitrary correlation time. Thus the generalization of some previous approximation schemes to two-noise driven flows presents a challenge.

With our focus being on two-noise driven transport over intervening fluctuating barriers we consider the archetypical situation of a fluctuating Ginzburg-Landau bistable dynamics, i.e.,

$$\dot{x}=ax-bx^3+g(x)\zeta(t)+h(x)\sqrt{2D}\,\xi(t)\ ,\qquad a>0\ ,\quad b>0\ . \tag{1.1}$$

Here, $\xi(t)$ denotes Gaussian white noise of strength D, with correlation $\langle\xi(t)\xi(s)\rangle=\delta(t-s)$ and $g(x)\zeta(t)$ models the presence of generally multiplicative colored noise. In the absence of colored noise and with additive white noise, i.e., $h(x)=1$, the flow in (1.1) reduces to the well-known Smoluchowski dynamics

$$\dot{x}=ax-bx^3+\sqrt{2D}\,\xi(t) \tag{1.2}$$

for which the escape time T is known exactly up to quadratures [8]. In the limit of weak noise, this escape time for a particle to leave the metastable state $x_\pm=\pm\sqrt{a/b}$ and to be trapped in the neighboring metastable state equals the well-known result [33]

$$T=\frac{\pi\sqrt{2}}{a}\left[1+\frac{3}{2}\frac{b}{a^2}D\right]\exp\left[\frac{a^2}{4bD}\right] \tag{1.3}$$

with $\Delta\Phi=(a^2/4b)$ being the Arrhenius energy (barrier height).

As a specific situation we shall investigate in this work the influence of additional parametric noise. In doing so we let the barrier curvature become a fluctuating quantity. Following Ref. [12] we set

$$a\to a+\zeta(t)\ , \tag{1.4}$$

where $\zeta(t)$ is assumed not to be correlated with the white noise source $\xi(t)$; i.e., $\langle\zeta(t)\xi(s)\rangle=0$. The stochastic dynamics in (1.3) thus becomes a process driven by two noise sources, one being additive and one being multiplicative, i.e.,

$$\dot{x}=ax-bx^3+x\zeta(t)+\sqrt{2D}\,\xi(t)\ . \tag{1.5}$$

With $a\to a+\zeta(t)$, the random curvature thus can assume both positive and negative values. Moreover, the barrier height becomes fluctuating; namely, $\Delta\Phi=0$, when $a+\zeta(t)<0$, and $\Delta\Phi=\{[a+\zeta(t)]^2/4b\}$, when $a+\zeta(t)>0$.

Up to this point, the noise statistics of $\zeta(t)$ with $\langle\zeta(t)\rangle=0$ has not been specified. Bearing in mind the central limit theorem we shall use throughout a Gaussian statistics for $\zeta(t)$. For our explicit considerations we use a Gaussian Markov process, namely, exponentially correlated Gaussian noise with correlation

$$\langle\zeta(t)\zeta(s)\rangle=\frac{Q}{\tau}\exp(-|t-s|/\tau)\ , \tag{1.6}$$

wherein τ denotes the noise correlation time. Our fluctuating double-well escape dynamics is then characterized by three parameters: The white intensity D, the colored noise intensity Q, where $R\equiv Q/D$ is held fixed, but otherwise arbitrary, and the noise correlation time τ.

The findings of our comprehensive study can be summarized as follows.

(i) The dynamics in (1.1) with (1.6) is equivalent to a two-dimensional Fokker-Planck process. With a general drift $ax-bx^3\to f(x)$ the corresponding (Stratonovitch) Langevin equations read [16]

$$\dot{x}=f(x)+g(x)\zeta(t)+h(x)\sqrt{2D}\,\xi(t)\ ,\qquad \dot{\zeta}=-\frac{\zeta}{\tau}+\frac{\sqrt{2Q}}{\tau}\eta(t)\ , \tag{1.7}$$

with $\{\eta(t),\xi(t)\}$ being *independent* Gaussian white noise forces with correlation

$$\langle\xi(t)\xi(0)\rangle=\langle\eta(t)\eta(0)\rangle=\delta(t)\ ,$$

and $\langle\xi(t)\eta(0)\rangle=0$.

(ii) Limit of white noise sources. With $\zeta(t)$ being white noise, (1.7) reduces to a one-dimensional Langevin equation driven by two white noise sources $\xi(t)$ and $\zeta(t)\to\sqrt{2Q}\,\eta(t)$. As is well known, the escape time T for a particle to leave the first metastable state at x_-, overcoming an instable transition state at x_0, $x_-<x_0<x_+$ and being trapped in the adjacent metastable state $x_+<x_-$, can then be expressed—via the mean first passage time $T^{\mathrm{MFPT}}(x=x_-\to x=x_+)=T(R)$—in closed form in terms of two quadratures [7,8]. With $R=Q/D$ finite, this white noise driven escape time $T(R)$ at weak noise $D\ll1$, $Q\ll1$ is always exponentially suppressed over the escape time in the Smoluchowski limit, see Eq. (1.3), i.e.,

$$T(R)<T(R=0)\equiv T\ . \tag{1.8}$$

Moreover, the decrease of $T(R)$ is Arrhenius-like and occurs *monotonically* with increasing R. Put differently, in accordance with the previous results in Refs. [10,12,13] the additional multiplicative noise source $g(x)\sqrt{2Q}\,\eta(t)$ assists the Smoluchowski escape dynamics by enhancing the effective temperature $D\to D(x)$ in a state-dependent way, i.e., if we set $h(x)=1$,

$$D\to D(x)=D+g^2(x)Q\geq D\ , \tag{1.9}$$

which consequently accelerates the escape dynamics.

(iii) Limit of small noise color. With small noise color $\tau\to0$ ("pink" noise), a consistent small τ expansion around the limit $\tau=0$ leads for the rate of change of the probability $\dot{p}_t(x)$ to a master equation that contains in leading order τ a third-order derivative. That is, contrary to common wisdom [23,34], there exists no consistent small-τ effective Fokker-Planck equation governing the dynamics of a multiplicative noise structure composed of two Gaussian noise sources with one being colored, cf. (1.5).

The next five points summarize the main results of the present work.

(iv) We construct novel approximation schemes for the nonlinear colored noise flow in (1.1) and (1.7). In doing so, we introduce the auxiliary, nonlinear process

$$\dot{u}=\zeta+(f/g)\left[1+\frac{Dh^2}{Qg^2}[1-\tau g(f/g)']\right]^{-1}\ , \tag{1.10}$$

which notably depends on the ratio of noise intensities D and Q. This process is ideally suited to obtain a one-dimensional Markov approximation for the asymptotic long-time properties of the non-Markovian Langevin equation in (1.1). In particular, we have with $\dot{u} \to 0$ as $\tau \to \infty$ the "large-τ approximation"

$$\dot{x} = \frac{f[1-\tau g(f/g)']}{[1+R(g/h)^2-\tau g(f/g)']} + h(x)\sqrt{2D}\,\xi(t)\ , \quad \tau \gg 1\ . \tag{1.11}$$

In the opposite limit $\tau \ll 1$, an adiabatic elimination of the fast process $\dot{u}$ yields a new, generalized unified colored noise approximation (UCNA) [21,28,29,32], given by the (Stratonovitch) Langevin equation

$$\dot{x} = [\tau\gamma(x,\tau)]^{-1}\{f(x)+g(x)\sqrt{2Q}\,\eta(t) + h(x)\sqrt{2D}\,\xi(t)\}\ , \tag{1.12}$$

with the effective friction $\gamma(x,\tau) \equiv \gamma(x,\tau;R=Q/D)$ defined in (5.3).

(v) These approximation schemes reduce to known results in the literature [21,28,29,32] in the limit $D \to 0$, i.e., when $R = Q/D \to \infty$. We now find that the generalized potential $\Phi(x,\tau)$ governing the exponential behavior of the stationary probability is exact for the linear case, cf. (4.1). Moreover, the effective Arrhenius energy for additive [i.e., $g(x)=h(x)=1$] colored noise driven escape in bistable, piecewise parabolic potentials with a cusp at the transition state x_0 is found to be exact.

(vi) The generalized potential $\Phi(x,\tau)$, as well as the escape time itself, become exact in the limit $\tau=0$ and $\tau \to \infty$. As a matter of fact the escape time is minimal at $\tau=0$, and monotonically increases, in Arrhenius-like manner, with increasing noise color towards the $\tau \to \infty$ limit, given by the Smoluchowski result in Eq. (1.3), i.e.,

$$T = T(R,\tau \to \infty) \geq T(R,\tau) \geq T(R,\tau=0) = T(R)\ . \tag{1.13}$$

(vii) Of special interest are the limiting behaviors at asymptotic values of the noise color τ and/or noise ratio R. Setting $h(x)=1$ one finds, for example, that

$$\frac{T(R,\tau \ll 1)}{T(R,\tau=0)} = \exp\left[\frac{R\tau^2}{D}\left\{-\int_{x_-}^{x_0} f\left[\left[\frac{gf}{1+Rg^2}\right]'\right]^2 dx\right\} + O(\tau^4)\right] \tag{1.14}$$

is enhanced proportional to τ^2. This behavior is in agreement with rigorous considerations taken from asymptotic path integral solutions of non-Markovian flows [22,32,35–37] when $\tau \to 0$, with $(\tau/D) \gg 1$. In contrast, the (*ad hoc*) truncated Fokker-Planck dynamics at second order for small τ, cf. (iii), predicts an exponential growth linear in τ.

(viii) For a constant, τ-independent variance $\langle \zeta^2 \rangle = Q$, i.e., $Q \to Q\tau$, the generalized UCNA in (1.12) still holds for finite, but small noise color $\tau < 1$. In contrast to the case with constant noise intensity, i.e., $Q = \int_0^\infty \langle \zeta(t)\zeta(0) \rangle dt$, cf. Eq. (1.6), this now implies that the Smoluchowski limit is assumed for $\tau=0$, i.e., $T(R \to R\tau, \tau=0) = T$. Our generalized UCNA predicts in this case an effective Arrhenius energy $\Delta\Phi(R \to R\tau,\tau)$ which *decreases* for increasing, small noise color, i.e.,

$$\Delta\Phi(R\tau,\tau) < \Delta\Phi(\tau=0) \quad \text{as } \tau \downarrow 0\ . \tag{1.15}$$

This fact is in accordance with very recent findings for the mean first passage time over fluctuating barriers [11(f), 11(g)].

II. FLUCTUATING DOUBLE WELL: LIMIT OF WHITE NOISE

We begin by elaborating in greater detail the case first discussed in Ref. [12]: the white noise driven overdamped parametric oscillator in (1.5). With $\tau \to 0$, we find from (1.6) that

$$\lim_{\tau \to 0} \langle \zeta(t)\zeta(s) \rangle = 2Q\delta(t-s)\ .$$

By use of the Stratonovitch interpretation for the stochastic differential equation in (1.5) we obtain a Fokker-Planck dynamics for the single event probability density $p_t(x)$, reading

$$\dot{p}_t = -\frac{\partial}{\partial x}\{(ax-bx^3+Qx)p_t\} + D\frac{\partial^2}{\partial x^2}\{(1+Rx^2)p_t\}\ . \tag{2.1}$$

By use of the dimensionless variables $x \to x(b/a)^{1/2}$, $t \to at$, $D \to Db/a^2$, and $Q \to Q/a$, Eq. (2.1) could be brought into a dimensionless form with $a=b=1$. We prefer, however, to stick to parameters carrying a dimension in order to exhibit explicitly the dependence on the potential parameters, and—most importantly—for useful (dimensional) checks of cumbersome calculations, e.g., see Eq. (2.9).

As is well known, the stationary probability $p(x)$ can readily be evaluated in terms of a generalized potential $\Phi(x)$ and a prefactor $K(x)$, i.e.,

$$\begin{aligned} p(x) &\equiv \left[\frac{K(x)}{Z}\right]\exp[-\Phi(x)/D] \\ &= \frac{Z^{-1}}{(1+Rx^2)^{1/2}}\exp\left\{\int_0^x \frac{ay-by^3}{D(1+Ry^2)}dy\right\} \\ &= \frac{Z^{-1}}{(1+Rx^2)^{1/2}}\exp\left\{\frac{-1}{2RD}\left[+bx^2-\left[a+\frac{b}{R}\right]\ln(1+Rx^2)\right]\right\}\ , \end{aligned} \tag{2.2}$$

where Z^{-1} is the normalization constant. We shall always assume that $p(x)$ is truly bistable; i.e., $p''(x_-)<0$, $p''(x_+)<0$, and $p''(x_0)>0$. This implies the restriction $Q=RD<a$; or $R<a/D$. With $D\to 0$, the ratio $R=Q/D$ stays finite when $Q\to 0$. With $\Phi'(x)=0$ at the stable states $x=x_\pm=\pm\sqrt{a/b}$, and at the unstable state $x_0=0$, the extrema of $\Phi(x)$ coincide with the deterministic steady states [38].

In passing, we note that in the absence of additive noise, i.e., $D=0$, the above model can be solved for its dynamics exactly ("Schenzle-Brand" model [20]): Its spectrum consists of a finite number of discrete eigenvalues and a continuum part. With $D=0$, it belongs to the class of generalized Verhulst models, studied extensively in the literature in the context of "noise-induced transition phenomena" [39,40].

Our prime concern is the study of the escape time $T(R)$. An escape occurs only for $D>0$. The mean first passage time (MFPT) to reach the state $x_+=+\sqrt{a/b}$, when a random walker originally started out at $x_-=-\sqrt{a/b}$, is given by the two quadratures [8]

$$T(R)=T_{\mathrm{MFPT}}(x_-\to x_+)$$
$$=D^{-1}\int_{x_-}^{x_+}\frac{dx}{(1+Rx^2)p(x)}\int_{-\infty}^{x}p(y)dy\ . \qquad (2.3)$$

With (2.2) the escape time has the structure

$$T(R)=D^{-1}\int_{x_-}^{x_+}dx\, H_1(x)\exp\left[\frac{\Phi(x)}{D}\right]\times\int_{-\infty}^{x}dy\, H_2(y)\exp\left[-\frac{\Phi(y)}{D}\right]. \qquad (2.4)$$

In our case we have $H_1(x)=H_2(x)=K(x)$.

For weak noise $D\ll 1$, this expression can be evaluated within the steepest descent approximation. By use of the formula, where $(\alpha,\beta,\gamma)\propto 1/D$, $\alpha\gg 1$,

$$\int_{-\infty}^{\infty}dx\,\exp(-\alpha x^2+\beta x^3+\gamma x^4)$$
$$\approx\int_{-\infty}^{\infty}(1+\beta x^3+\gamma x^4+\tfrac{1}{2}\beta^2x^6)\exp(-\alpha x^2)dx$$
$$=\left[\frac{\pi}{\alpha}\right]^{1/2}\left[1+\frac{3}{4}\frac{\gamma}{\alpha^2}+\frac{15}{16}\frac{\beta^2}{\alpha^3}\right], \qquad (2.5)$$

the result for $T(R)$—up to order $O(D^2)$—is explicitly given by

$$T(R)=\exp(\Delta\Phi/D)\frac{2\pi H_1(x_0)H_2(x_-)}{[|\Phi''(x_0)|\Phi''(x_-)]^{1/2}}\times\Bigg\{1+D\Bigg[\frac{1}{2}\left[\frac{H_1''(x_0)}{H_1(x_0)|\Phi''(x_0)|}+\frac{H_2''(x_-)}{H_2(x_-)\Phi''(x_-)}\right]+\frac{1}{8}\left[\frac{\Phi''''(x_0)}{|\Phi''(x_0)|^2}-\frac{\Phi''''(x_-)}{[\Phi''(x_-)]^2}\right]$$
$$+\frac{1}{2}\left[\frac{H_1'(x_0)\Phi'''(x_0)}{H_1(x_0)|\Phi''(x_0)|^2}-\frac{H_2'(x_-)\Phi'''(x_-)}{H_2(x_-)[\Phi''(x_-)]^2}\right]$$
$$+\frac{5}{24}\left[\frac{(\Phi'''(x_0))^2}{|\Phi''(x_0)|^3}+\frac{[\Phi'''(x_-)]^2}{[\Phi''(x_-)]^3}\right]\Bigg]\Bigg\}. \qquad (2.6)$$

This explicit steepest descent result for Eq. (2.4)—valid for multiplicative noise—will prove useful throughout the remainder of this work. With $H_1(x)=H_2(x)=\mathrm{const}$, the first and third contributions inside the square bracket, multiplied by D, vanish and the result in Eq. (2.6) reduces to the well-known expression for the Smoluchowski escape time, corrected for first-order temperature effects [41]; see (1.3).

The Arrhenius energy $\Delta\Phi$ is evaluated to read

$$\Delta\Phi(R)=\int_0^{\sqrt{a/b}}\frac{ay-by^3}{(1+Ry^2)}dy$$
$$=(2R)^{-1}\left\{-a+\left[a+\frac{b}{R}\right]\ln\left[1+R\frac{a}{b}\right]\right\}$$
$$<\Delta\Phi(R=0)\ . \qquad (2.7)$$

In agreement with Ref. [12], the escape time is always exponentially reduced over the case $T(R=0)$, i.e., $T(R=0)>T(R\neq 0)$. For $R\to 0$ the barrier height $\Delta\Phi(R)$ approaches

$$\Delta\Phi(R)=\frac{a^2}{4b}\left[1-\frac{1}{3}R\frac{a}{b}+O(R^2)\right],\quad R\to 0\ . \qquad (2.8)$$

From (2.7) it also follows that $\Delta\Phi(R)$ monotonically *decreases* with *increasing* R. The escape $T(R)$ itself follows from (2.6), including the correction of order D, as

$$T(R)=\frac{\pi\sqrt{2}}{a}\exp\left[\frac{\Delta\Phi(R)}{D}\right]\times\left\{1+\left[\frac{Db}{a^2}\right]\left[1+R\frac{a}{b}\right]^{-1}\times\left[\frac{3}{2}+3R\frac{a}{b}+\frac{17}{12}\left[R\frac{a}{b}\right]^2\right]\right\}. \qquad (2.9)$$

III. FLUCTUATING DOUBLE WELL: LIMIT OF SMALL NOISE CORRELATION TIME

With a good understanding of our model in the limit of white driving noise forces we are courageous enough to engage with the more realistic situation of colored parameter fluctuations $\zeta(t)$. Our objective is to investigate the influence of colored noise for the barrier fluctuations in analytical terms. This task, however, encounters a major difficulty, namely, the problem of a nonlinear stochastic flow which is non-Markovian. Here, we shall focus first on the behavior of escape in the presence of fluctuating potential barriers near the limit of white noise. With exponentially correlated Gaussian noise in mind we shall treat Eq. (1.5) in the limit of small noise color, i.e., $a\tau \ll 1$. The authors have not been able to obtain analytical results for the escape time of the corresponding two-dimensional Markovian flow in (1.7), which clearly does not obey detailed balance. To make some progress we invoke an approximation scheme. With our interest being in finding *analytical* approximations for the escape time we seek an approximation procedure which reduces to a one-dimensional, effective Fokker-Planck equation.

To start, let us imagine that the additive white noise $\xi(t)$ is switched "off." Then we deal with a one-dimensional, multiplicative colored noise flow for which one can construct an effective Fokker-Planck equation by expanding the functional derivative $\delta x(t)/\delta\zeta(s)$ around the white noise limit [16,24,26–28,30,31,42,43], i.e., for $D=0$ the rate of change of the probability $p(x;D=0;t)\equiv\tilde{p}(x,t)$ reads

$$\dot{\tilde{p}}_t=-\frac{\partial}{\partial x}\{[ax-bx^3+Qx(1-2\tau bx^2)]\tilde{p}_t\}+Q\frac{\partial^2}{\partial x^2}\{[x^2(1-2\tau bx^2)]\tilde{p}_t\} . \quad (3.1)$$

Switching back "on" the neglected white noise $\xi(t)$ we thus obtain an approximation for the rate of change of the probability p_t reading [23,34]

$$\dot{p}_t=-\frac{\partial}{\partial x}\{[ax-bx^3+Qx(1-2\tau bx^2)]p_t\}+D\frac{\partial^2}{\partial x^2}\{[1+Rx^2(1-2\tau bx^2)]p_t\} . \quad (3.2)$$

With (3.2) we can evaluate the escape time just as with the white noise case in Sec. II. The result reads in leading order

$$T(R,\tau)=\frac{\sqrt{2\pi}}{a}\exp\left[2Rb\tau\int_0^{\sqrt{a/b}}\frac{y^3dy}{[1+Ry^2(1-2\tau by^2)]}\right]\times\exp\left[\frac{\Delta\Phi(R,\tau)}{D}\right] , \quad (3.3)$$

with an effective Arrhenius factor

$$\Delta\Phi(R,\tau)=\int_0^{\sqrt{a/b}}\frac{ay-by^3}{[1+Ry^2(1-2\tau by^2)]}dy . \quad (3.4)$$

With $a\tau\ll 1$, this reduces to

$$\Delta\Phi(R,\tau)=\Delta\Phi(R)+2\tau bR\int_0^{\sqrt{a/b}}\frac{(ay-by^3)y^4}{(1+Ry^2)^2}dy \geq \Delta\Phi(R,\tau=0) . \quad (3.5)$$

Here, the second contribution in Eq. (3.5) yields a positive value. Thus we find that the escape time exponentially *increases* with increasing correlation time proportional to τ, i.e., we have

$$T(R,\tau)>T(R,\tau=0) , \quad (3.6)$$

with the increase being Arrhenius-like.

The above analysis is just fine if we had not made a mistake: We obtained (3.2) within a two-step procedure with the white noise first switched off—and then switched on again. The *simultaneous* presence of the two Gaussian noise sources, however, with differing correlation time (i.e., $\tau_\zeta=\tau$ versus $\tau_\xi=0$) adds additional contributions to the rate of change of p_t [30,31]. Following the reasoning of Dekker [31] we find that (3.2) becomes augmented by a non-Fokker-Planck contribution [16,30,31], i.e., a third-order term in (∂/∂_x), which in our case, see Ref. [44], is explicitly evaluated as

$$2\tau QD\left\{\frac{\partial}{\partial x}x\frac{\partial^2}{\partial x^2}p_t\right\} . \quad (3.7)$$

It explicitly involves the product QD, thereby reducing to (3.1) for $D=0$. Hence the result in (3.3) cannot be trusted quantitatively. The general conclusion, however, that colored noise increases the escape time with increasing noise color should not be affected by the non-Fokker-Planck contribution: This conclusion is consistent with the result of mean-field decoupling theory [25], predicting that the effective colored diffusion $D_{\text{color}}<D$(zero color) [25–27]; whence slowing down the escape dynamics.

Before we proceed we emphasize that the leading contribution in (3.7) rules out the construction of a consistent effective Fokker-Planck equation, as presented with (3.2); a fact which is not sufficiently appreciated in the literature [23,34]. Hence, use of (3.2)—although appealing—can generally be justified only *a posteriori*.

IV. TWO-NOISE DRIVEN LINEAR FLOW

Before we elucidate the approximation schemes for arbitrary noise color we investigate the exactly solvable linear colored noise flow

$$\dot{x}=-ax+\zeta(t)+\sqrt{2D}\,\xi(t) . \quad (4.1)$$

This linear flow follows by setting $b=0$ in (1.5), and using a stable parabolic potential $a\to -a<0$. The corresponding two-dimensional Markovian flow, i.e.,

$$\dot{x}=-ax+\zeta(t)+\sqrt{2D}\,\xi(t) ,\qquad \dot{\zeta}=-\frac{\zeta}{\tau}+\frac{\sqrt{2Q}}{\tau}\eta(t) , \quad (4.2)$$

presents a two-dimensional Gauss-Markov process. Its exact stationary probability is a Gaussian, reading

$$p(x,\zeta)=Z^{-1}\exp(\lambda y^2)\exp\left\{\left[\alpha-\left[\frac{\beta}{2\lambda}\right]^2\right]x^2\right\}, \quad (4.3a)$$

where

$$y\equiv\zeta+\left[\frac{\beta}{2\lambda}\right]x=\zeta+\frac{-ax}{1+(D/Q)(1+\tau a)} \quad (4.3b)$$

and

$$\alpha=-\tfrac{1}{2}(a+\tau^{-1})\beta\,,$$
$$\beta=a(a+\tau^{-1})[D(a+\tau^{-1})^2+Q\tau^{-2}]^{-1}\,, \quad (4.3c)$$
$$\lambda=-\frac{\tau^2}{2Qa}\beta[\tau^{-1}D(a+\tau^{-1})+\tau^{-2}Q]\,.$$

Integrating over ζ yields the exact, colored noise dependent stationary probability for x, i.e.,

$$p(x,\tau)=\left[\frac{2\pi\{D(1+\tau a)+Q\}}{a(1+\tau a)}\right]^{-1/2}\times\exp\left\{\frac{-x^2a(1+\tau a)}{2[D(1+\tau a)+Q]}\right\}. \quad (4.4)$$

We note from (4.3a) that the pair (x,ζ) is correlated. In distinct contrast, the transformed process $(x,\zeta)\to(x,y)$ is correlation free. As demonstrated previously [28], the choice of the correlation-free process is a necessary prerequisite to approximate accurately the stationary probability of a higher-dimensional Markovian flow—via an adiabatic elimination—by a reduced one-dimensional Markovian flow. In the next section we seek a colored noise approximation scheme which correctly reproduces the stationary probability in (4.4). This constitutes a truly nontrivial task: All of the presently available colored noise approximation procedures such as the small-τ theories [24–27,30,31,42–44], the decoupling theory [25], the unified colored noise scheme in Ref. [21(b)], or perturbative path-integral studies at small noise color [22,32] fail to reproduce the exact result in (4.4) for this two-noise driven linear case.

V. MARKOV APPROXIMATION FOR PARAMETRICALLY DRIVEN COLORED NOISE FLOWS

As made obvious within the study of the white noise limit in Sec. II and the small-τ limit in Sec. III the exponential leading part of the escape time at weak noise is dominantly ruled by the generalized potential $\Phi(x,\tau)$ of the stationary probability

$$p(x,\tau)=K(x,\tau)\exp[-\Phi(x,\tau)/D]\,.$$

Our main focus thus will consist in seeking an approximation for the long-time dynamics of the *nonlinear* one-dimensional non-Markovian process in Eq. (1.1), or equivalently the long-time dynamics of the enlarged two-dimensional Markov dynamics in Eq. (1.7). In doing so we seek a one-dimensional Markov process whose stationary long-time properties, such as the generalized potential (but not necessarily its short-time dynamics), is close to the long-time properties of the original non-Markovian process. We start from the general, two-dimensional (Stratonovitch) stochastic differential equation in (1.7). We next introduce the auxiliary nonlinear process $\dot{u}(t)$, defined by

$$\dot{u}=\zeta+\frac{(f/g)}{1+(Dh^2/Qg^2)[1-\tau g(f/g)']}\,. \quad (5.1)$$

Here the prime denotes a differentiation after x. With additive colored noise, i.e., $h(x)=g(x)=1$, this process $\dot{u}$ coincides for $f(x)=-ax$ with the uncorrelated process y introduced in (4.3b). Moreover, setting $D=0$, the process $\dot{u}(t)$ yields the previously established unified colored noise approximation for colored noise driven one-dimensional stochastic flows [21,28,29,32]. The dependence on the multiplicative noise functions $g(x)$ and $h(x)$ for the auxiliary process $\dot{u}(t)$ in (5.1) is fixed *uniquely* by noting that a nonlinear transformation of the linear process $x(t)$ in Sec. IV, i.e., $x(t)\to F(x(t))$ yields a (Stratonovitch) Langevin equation with multiplicative noise sources whose exact stationary probability coincides with the correspondingly transformed stationary probability. It is remarkable, however, that with $D\neq 0$ the auxiliary process $\dot{u}(t)$ depends on the noise intensities D and Q. After a lengthy but straightforward calculation we can recast the two-dimensional flow in Eq. (1.7) as

$$\dot{x}=g\dot{u}-\frac{(1-A)}{A}g(f/g)+h(x)\sqrt{2D}\,\xi\,, \quad (5.2a)$$

where

$$A=1+\left[\frac{Dh^2}{Qg^2}\right]\left[1-\tau g(f/g)'\right], \quad (5.2b)$$

and an exact Langevin equation for $\dot{u}$, i.e.,

$$\ddot{u}=\dot{\zeta}+\left[\frac{(f/g)}{A}\right]'\dot{x}$$
$$=-\gamma(x,\tau)\dot{u}+\frac{(f/g)}{A}\left[\tau^{-1}+g(1-A)\left[\frac{f/g)}{A}\right]'\right]$$
$$+\frac{\sqrt{2Q}}{\tau}\eta+\left[\frac{f/g}{A}\right]'h(x)\sqrt{2D}\,\xi\,. \quad (5.2c)$$

Note that (5.2c) is linear in $\dot{u}$. This feature is nontrivial and results solely due to the ingenious choice for the auxiliary process $\dot{u}$ in (5.1) The effective friction $\gamma(x,\tau)$ is given as

$$\gamma(x,\tau)=\left\{[\tau^{-1}-g(f/g)']\left[1+\frac{Dh^2}{Qg^2}\right]+fA'/A\right\}\Big/A\,. \quad (5.3)$$

The form of this effective friction for the case of a fluctuating double well, cf. Eq. (1.5), is depicted in Fig. 1 for $\tau=0.1$ and 1. We note that $\gamma(x,\tau)$ is of order τ^{-1} throughout the bistable region. Without loss of generality, we use in the following—for the sake of simplicity only—additive white noise, i.e., we set $h(x)=1$.

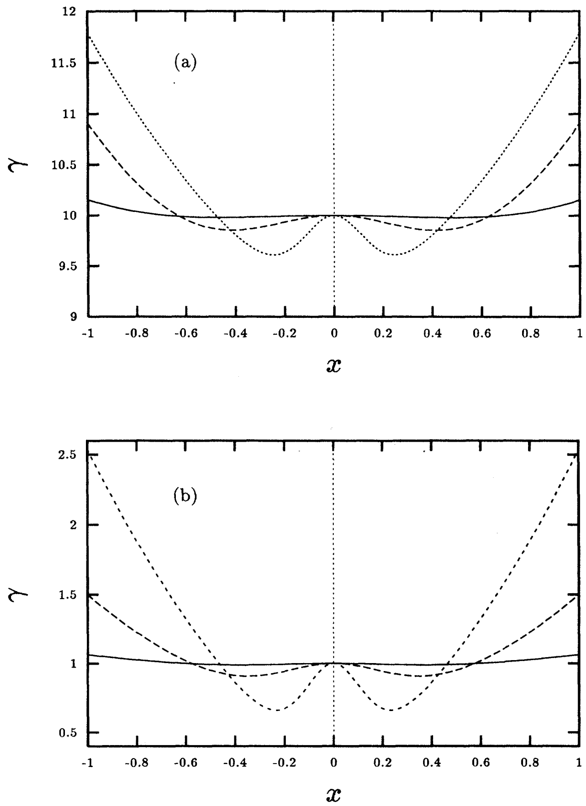

FIG. 1. The effective friction for the fluctuating double well in Eq. (1.5) with $a=b=1$ is depicted for different noise ratios $R=Q/D$. In (a) we use $\tau=0.1$ while in (b) we have $\tau=1$. The solid line refers to $R=0.1$, the dashed line to $R=1$, and the dotted line to $R=10$.

A. Markovian approximation for large noise color

For large noise color $\tau \gg 1$, the process $\dot{u}(t)$ approaches $\zeta(t)$. With $\zeta(t)$ given in Eq. (1.6), the integrated noise correlation

$$S_\zeta(\omega=0)=\frac{2Q}{\tau}\int_0^\infty \exp(-t/\tau)dt=2Q \tag{5.4}$$

remains a constant, and $\langle \zeta^2 \rangle$ approaches zero proportionally to τ^{-1}. Therefore, with $\dot{u}(t)\to 0$ as $\tau\to\infty$, we find from (5.2a) and (5.2b) the large-τ Markov approximation, reading

$$\dot{x}=f\frac{1-\tau g(f/g)'}{[1+Rg^2-\tau g(f/g)']}+\sqrt{2D}\,\xi(t) \ . \tag{5.5}$$

With $h(x)$ not a constant, the corresponding result reads as in (1.11).

The stationary probability is readily evaluated, i.e.,

$$p(x,\tau)=Z^{-1}\exp[-\Phi_L(x,\tau)/D] \ , \tag{5.6}$$

with the generalized potential for large (L) noise color given by

$$\Phi_L(x,\tau)=-\int_0^x \frac{f[1-\tau g(f/g)']}{1+Rg^2-\tau g(f/g)'}dy \ . \tag{5.7}$$

For the stochastic differential equation (SDE) in Eq. (5.5), the escape time $T(x_-\to x_+)$ is given by the MFPT

$$T(R,\tau)=D^{-1}\int_{x_-}^{x_+}\frac{dx}{p(x,\tau)}\int_{-\infty}^{x}p(y,\tau)dy \ , \tag{5.8}$$

with $p(x,\tau)$ given in (5.6). For weak noise $D \ll 1$, this escape time is up to order $O(D)$ evaluated as

$$T(R,\tau\gg 1)=\frac{2\pi}{[|\Phi_L''(x_0;R,\tau)|\Phi_L''(x_-,R,\tau)]^{1/2}} \times\exp(\Delta\Phi_L(R,\tau)/D) \ , \tag{5.9}$$

with the (effective) Arrhenius energy given by

$$\Delta\Phi_L(R,\tau)=-\int_{x_-}^{x_0}\frac{f[1-\tau g(f/g)']}{1+Rg^2-\tau g(f/g)'} \ . \tag{5.10}$$

Next we discuss exactly solvably limits. (i) For a linear flow with $f(x)=-ax$ and $g(x)=\text{const}$, the stationary probability in Eqs. (5.6) and (5.7) coincides precisely with the exact result in (4.4). (ii) For $\tau\to\infty$, the above results smoothly converge to the Smoluchowski limit in Eq. (1.3), being consistent with the SDE in (5.5), when $\tau\to\infty$. (iii) For $R\to 0$, i.e., $Q\to 0$, one again recovers the correct Smoluchowski behavior in (1.2). (iv) With $D\equiv 0$, i.e., $R\to\infty$, the generalized potential in (5.6) and (5.7) coincides with the UCNA theory elucidated previously by several authors [21,28,29,32].

Finally we find that with $-g(f/g)'\geq 0$ within $[x_-,x_0]$ the Arrhenius energy obeys

$$\Delta\Phi(R=0)\geq\Delta\Phi_L(R,\tau)\geq\Delta\Phi_L(R,\tau=0) \ , \tag{5.11}$$

which monotonically increases with increasing noise color τ towards the Smoluchowski result $\Delta\Phi(R=0)=a^2/4b$, cf. (1.3).

Although the approximation for large noise color $\tau\gg 1$ in (5.5)–(5.7) yields the exact result for the generalized potential of the stationary probability (but not its prefactor) even for the case $\tau=0$—and thus naturally bridges the result from $\tau=0$ up to $\tau\to\infty$—the approximation is not expected to represent accurate results for very small τ values, where $\dot{u}(t)\neq 0$. Hence we shall study the behavior at small noise color separately.

B. Markovian approximation for small noise color

Starting from the exact dynamics in (5.2) we observe that the effective friction $\gamma(x,\tau)$ in (5.3) approaches very large values as $\tau\to 0$. With the new time $t\to t/\sqrt{\tau}$ we adiabatically eliminate (i.e., we set $\ddot{u}=0$) the fast process $\dot{u}(t)$ in (5.2c) in regions of x values that guarantee a positive effective friction; i.e., in regimes of x values where $[1-\tau g(f/g)']>0$. Solving from (5.2c) for the stationary process $\dot{u}(t)$ and substituting this adiabatic approximation into (5.2a) yields a one-dimensional Markovian description for the non-Markovian flow in (1.1) with a general drift $f(x)$. The resulting (Stratonovitch) SDE is evaluated to read

$$\dot{x}=\frac{1}{\tau\gamma(x,\tau)}[f(x)+\sqrt{2Q}\,g(x)\eta(t)+\sqrt{2D}\,\xi(t)] \ , \tag{5.12}$$

with its straightforward generalization for nonconstant

$h(x)$ given in (1.12).

Its stationary probability at small (S) noise color $\tau<1$ is readily evaluated, i.e.,

$$p(x,\tau)=\frac{Z^{-1}}{|D_{\rm eff}(x,\tau)|^{1/2}}\exp[-\Phi_S(x,\tau)/D]\ , \tag{5.13}$$

where

$$D_{\rm eff}(x,\tau)=D[1+Rg^2(x)]/[\tau\gamma(x,\tau)]^2\ . \tag{5.14}$$

The generalized potential is given by

$$\Phi_S(x,\tau)=-\int_0^x\frac{f[1-\tau g(f/g)']dy}{1+Rg^2-\tau g(f/g)'}-\tau\int_0^x\frac{Rg^2f^2\{-(2g'/g)+\tau[g'(f/g)'-g(f/g)'']\}dy}{(1+Rg^2-\tau g(f/g)')^2(1+Rg^2)} \tag{5.15a}$$

$$\equiv\Phi_1(x,\tau)+\Phi_2(x,\tau)=\Phi_L(x,\tau)+\Phi_2(x,\tau)\ , \tag{5.15b}$$

where $\Phi_2(x,\tau)$ is given by the second contribution on the right-hand side r.h.s. of (5.15a). From (5.15a) and (5.15b) we observe that $\Phi_1(x,\tau)$ coincides precisely with the generalized potential of the large-τ approximation in Eq. (5.7). The contribution $\Phi_2(x,\tau)$ vanishes for $\tau=0$.

The properties of this adiabatic approximation are as follows. (i) With $\tau\to0$, we recover of course the correct white noise limit in Sec. II. (ii) For a linear flow we find that $\Phi_2(x,\tau)$ vanishes identically, i.e., $p(x,\tau)$ yields again the *exact* result in Eq. (4.4). (iii) Setting $R=0$, we again find the correct white noise (Smoluchowski) limit. In particular, $\Phi_1(x,\tau;R=0)$ does not depend on noise color τ—as indeed should be the case. (iv) With $D\equiv0$, we recover the UCNA theory [28,21,29,32].

Next we compare this small noise color approximation for the stationary dynamics with the small-τ approximation put forward in Sec. III. In doing so we shall concentrate on the Arrhenius energy

$$\Delta\Phi_S(R,\tau)=\Phi_S(x_0,\tau)-\Phi_S(x_-,\tau)\ . \tag{5.16}$$

From (5.15a) one finds to lowest order in τ

$$\Delta\Phi_S(R,\tau)=\Delta\Phi(R,\tau=0)+\frac{\tau R}{2}\int_{x_-}^{x_0}\left\{\frac{f^2g^2}{(1+Rg^2)^2}\right\}'dx+O(\tau^2) \tag{5.17}$$

which with $f(x_0)=f(x_-)=0$ results in

$$\Delta\Phi_S(R,\tau)=\Delta\Phi(R,\tau=0)+O(\tau^2)\ . \tag{5.18}$$

This behavior differs from the small-τ theory in Eq. (3.5). Which result is more accurate? The systematic adiabatic approach is certainly more convincing in obtaining a good approximation for the generalized potential, since it does not involve the procedure of neglecting in an *ad hoc* manner the non-Fokker-Planck contribution in Eq. (3.7). Interestingly enough, however, the contribution in Eq. (3.5) being proportional to τ coincides precisely with the linear contribution of $\Delta\Phi_1(R,\tau)$. This latter contribution is exactly canceled by the linear contribution stemming from $\Delta\Phi_2(R,\tau)$.

This result that the small-τ correction for the effective Arrhenius energy $\Delta\Phi_S(R,\tau)$ starts with τ^2 rather than with τ can be rigorously proved by studying the Arrhenius energy as follows from a study of the corresponding path-integral expression for the stationary probability [22,32,35–37]: Expanding the action around the $\tau=0$ solution—in the limit $\tau\to0$, but with $(\tau/D)\to\infty$—indeed yields for the action between two fixed points $f(x_-)=f(x_0)=0$ a contribution starting proportional to τ^2 [22] rather than proportional to τ.

Before proceeding let us look in closer detail at the contribution $O(\tau^2)$ Setting $g(x)=1$, the τ^2 contribution follows from $\Phi_S(x,\tau)$ as

$$O(\tau^2)=\frac{\tau^2R}{(1+R)^3}\left\{\int_{x_-}^{x_0}f(f')^2dy+\int_{x_-}^{x_0}f^2f''dy\right\} \tag{5.19a}$$

$$=\frac{R\tau^2}{(1+R)^3}\int_{x_-}^{x_0}f(ff')'dy\ . \tag{5.19b}$$

With x_0 a fixed point, i.e., $f(x_-)=f(x_0)=0$, one finds for the Arrhenius energy, after a partial integration,

$$\Delta\Phi(R,\tau)=\Delta\Phi(R,\tau=0)-\frac{R\tau^2}{(1+R)^3}\int_{x_-}^{x_0}f(f')^2dy\ , \tag{5.20}$$

with the correction being positive valued. We observe that the $O(\tau^2)$ contribution in Eq. (5.20) coincides precisely with the *negative* of the first contribution in Eq. (5.19a), stemming from $\Phi_1(x,\tau)$. Note that for a linear flow with $f''(x)=0$ the τ^2 contribution in Eq. (5.19) is determined solely by the—in this case fully exact—first contribution in Eq. (5.19a). In contrast to the case with two fixed points, the effective Arrhenius energy $\Delta\Phi(R,\tau)$ starts in this case, however, proportional to τ, cf. Eq. (5.17).

Yet another crucial test for our theory is provided by the bistable potential $U(x)$ composed of piecewise parabolic wells with a cusp at $x=x_0$. With $x_-=0$, $x_+=2$, and $x_0=1$ we set

$$U(x)=\begin{cases}\frac{a}{2}x^2\ , & x\le1\\ \frac{a}{2}(x-2)^2\ , & x\ge1\ .\end{cases} \tag{5.21a}$$

With $g(x)=1$ we find for arbitrary noise color τ

$$\Delta\Phi_S(R,\tau)=\Delta\Phi_L(R,\tau)=\frac{a(1+\tau a)/2}{1+R+\tau a}\ . \tag{5.21b}$$

This result fully agrees with the exact result derived recently by Reimann [45].

VI. ESCAPE IN A FLUCTUATING DOUBLE WELL

The general theory of Sec. V is next applied to the archetype flow of a fluctuating double well in Eq. (1.5).

A. Deterministic dynamics

With (1.5) the deterministic dynamics follows from the SDE in Eq. (1.7) by setting $Q=D=0$, i.e.,

$$\dot{x}=ax-bx^3+x\zeta\ ,\qquad \dot{\zeta}=\frac{1}{\tau}\zeta\ . \tag{6.1}$$

In Figs. 2(a)–2(c) we depict this deterministic dynamics for various values of the noise color τ. With $g(x)=x=-g(-x)$, $f(x)=ax-bx^3=-f(-x)$, the flow obviously exhibits a reflection symmetry about the ζ axis. Thus the separatrix is given by the line $x=0$. The flow lines exhibit turning points characterized by $d\zeta/dx=\infty$. Setting

$$\dot{x}=\frac{dx}{d\zeta}\left[\frac{d\zeta}{dt}\right]=-\frac{1}{\tau}\zeta\left[\frac{dx}{d\zeta}\right]\ , \tag{6.2}$$

the sequence of turning points $d\zeta/dx=\infty$ (or $dx/d\zeta=0$) from a curve $s(x)$, i.e.,

$$s(x)=-a+bx^2\ . \tag{6.3}$$

Most importantly we note that the stochastic dynamics in the *absence* of white noise $\xi(t)$ (i.e., $D=0$) *cannot* cross the separatrix line $x=0$; i.e., no escape from $x_-\to x_+$ (and vice versa) can occur. This is so because the $\eta(t)$ fluctuations are acting solely along the ζ axis.

B. Generalized potential

The stationary probability follows for large τ directly from Eq. (5.5) where the diffusion is constant. For small

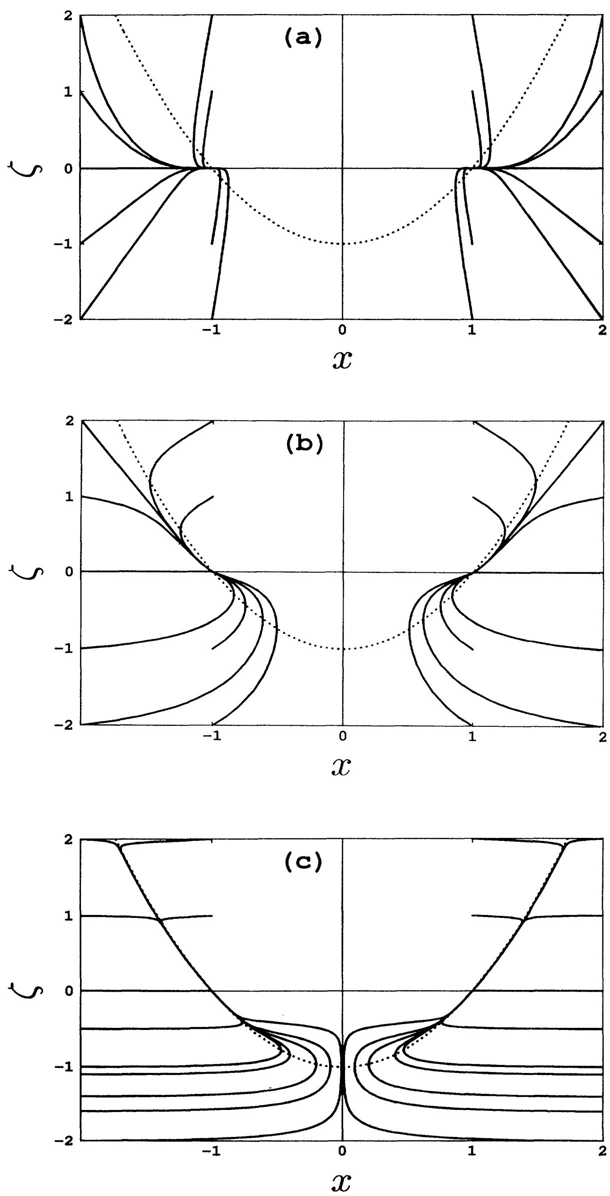

FIG. 2. Deterministic trajectories of the system in Eq. (6.1) written with the dimensionless variables, i.e., $x\to(b/a)^{1/2}x$, $\zeta\to\zeta/a$, $t\to at$, $\tau\to a\tau$, so that $\dot{x}=x-x^3+x\zeta$, $\dot{\zeta}=-\zeta/\tau$. (a)–(c) are for different strengths of noise color: (a) $\tau=0.1$, (b) $\tau=1$, and (c) $\tau=15$. The dotted curve gives the line of turning points $y(x)=-1+x^2$ in Eq. (6.3), where $(d\zeta/dx)=\infty$.

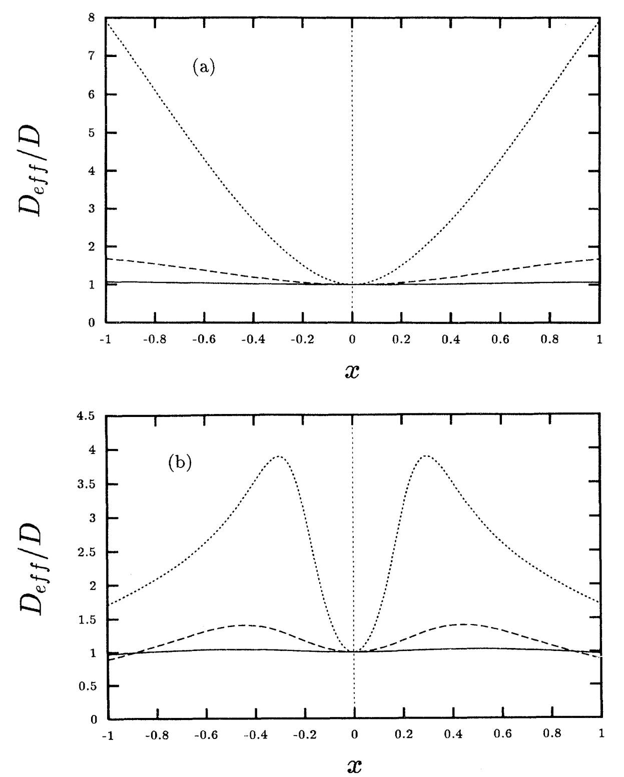

FIG. 3. The normalized effective diffusion $D_{\rm eff}(x,\tau)/D$ Eq. (6.4) with $a=b=1$ is depicted for $\tau=0.1$ (a) and $\tau=1$ (b), and different noise ratios $R=0.1$ (solid), $R=1$ (dashed), and $R=10$ (dotted).

noise color the corresponding (Stratonovitch) SDE in Eq. (5.12) has a state-dependent diffusion, cf. Eq. (5.14), i.e. with $R=Q/D$ and $-g(f/g)'=2b\tau x^2>0$ one finds

$$\begin{aligned} D_{\text{eff}}(x,\tau) &= D(1+Rx^2)[\tau\gamma(x;\tau,R)]^{-2} \\ &= D(1+Rx^2)\left[\frac{(1+2\tau bx^2)(1+Rx^2)}{[1+(R+2\tau b)x^2]} - \frac{2\tau Rx^2(a-bx^2)}{[1+(R+2\tau b)x^2]^2}\right]^{-2}, \end{aligned} \tag{6.4}$$

obeying $D_{\text{eff}}(x=0,\tau)/D=1$.

The condition $[2\tau b+R(1-a\tau)]\geq 0$ yields a sufficient criterion for the bracket in (6.4) to assume finite, strictly positive values for all x values. This latter condition guarantees the positivity of the effective friction $\gamma(x,\tau)$ used within the adiabatic elimination procedure, cf. Fig. 1. With $a\tau\leq 1$, it is obeyed for all values of R. Hence the effective diffusion in (6.4) is positive, and finite, over the whole bistable region $[x_-,x_+]$, $x_\pm=\pm\sqrt{a/b}$, cf. Fig. 3.

For the fluctuating double well the generalized potentials $\Phi_L(x,\tau)$ and $\Phi_S(x,\tau)$ can be expressed in analytical closed form. For example, one finds

$$\begin{aligned} \Phi_L(x,\tau)=\Phi_1(x,\tau) &= -\frac{b}{2}x^2\frac{2\tau b+(2\tau a-1)(R+2\tau b)}{(R+2\tau b)^2}+\frac{x^4\tau b^2}{2(R+2\tau b)} \\ &\quad -\frac{1}{2(R+2\tau b)^3}[a(R+2\tau b)^2-b(2a\tau-1)(R+2\tau b)-2\tau b^2]\ln[1+x^2(R+2\tau b)] . \end{aligned} \tag{6.5}$$

Of practical interest is the effective Arrhenius energy $\Delta\Phi(R,\tau)=\Phi(x_0,\tau)-\Phi(x_-,\tau)$. In terms of $Z\equiv(a/b)R+2a\tau$ one obtains, cf. Fig. 4,

$$\begin{aligned} \Delta\Phi_L(R,\tau)=\Delta\Phi_1(R,\tau) &= \frac{a^2}{2b}\left\{\frac{2a\tau+Z(2a\tau-1)}{Z^2}-\frac{a\tau}{Z}+\frac{Z^2-(2a\tau-1)Z-2a\tau}{Z^3}\ln(1+Z)\right\} \\ &= \Delta\Phi(R,\tau=0)+2\tau Rb\int_0^{\sqrt{a/b}}\frac{(ax-bx^3)x^4}{(1+Rx)^2}dx+O(\tau^2) \end{aligned} \tag{6.6}$$

and

$$\Delta\Phi_S(R,\tau)=\Delta\Phi_L(R,\tau)+\frac{a^3R}{2b^2}\left\{\frac{(1+Z)}{Z^2}-\frac{2\tau b}{Z^2R}-\frac{(Z^3-2ZaR/b+4Z^2+3Z-2aR/b)}{2a\tau Z^3}\ln(1+Z) + \frac{[1+(a/b)R]^2}{2a\tau(Ra/b)^2}\ln\left[1+\frac{a}{b}R\right]\right\} \tag{6.7a}$$

$$=\Delta\Phi(R,\tau=0)+0\times\tau+O(\tau^2) . \tag{6.7b}$$

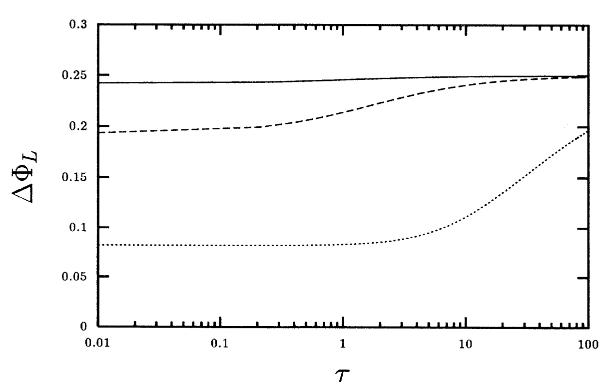

FIG. 4. The effective Arrhenius energy $\Delta\Phi_L(R,\tau)$ with $a=b=1$ is depicted for different values of $R=0.1$ (solid), $R=1$ (dashed), and $R=10$ (dotted). Note that the Arrhenius energy monotonically rises towards the limiting Smoluchowski value $\Delta\Phi_L=\frac{1}{4}$ as $\tau\rightarrow\infty$.

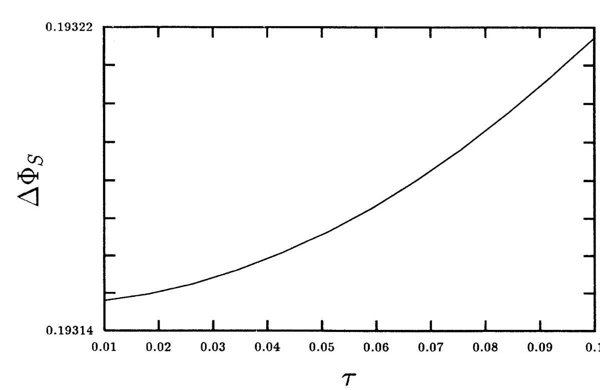

FIG. 5. The exponential part of the escape time is governed at small noise color by the effective Arrhenius energy $\Delta\Phi_S$ in Eq. (6.7). Using $a=b=1$ and a noise ratio $R=1$ the effective Arrhenius energy $\Delta\Phi_S$ clearly depicts initial quadratic growth with increasing noise color τ.

The behavior of Eq. (6.7) versus small noise color τ is depicted in Fig. 5.

The fact that the potential $\Phi_L(x,\tau)$ approaches the correct limit at both $\tau=0$ and $\tau\to\infty$ allows one to construct a smooth crossover approach. From the limiting behavior of the diffusion coefficient $D_{\text{eff}}(x,\tau)$ in Eq. (5.5) and Eq. (6.4), respectively, i.e.,

$$D_{\text{eff}}(x,\tau)=\begin{cases} D & \text{if } R\to 0\ , \\ D(1+x^2R) & \text{if } \tau\to 0\ , \\ D & \text{if } \tau\to\infty\ , \end{cases} \tag{6.8}$$

we set for the crossover theory the (Stratonovitch) SDE

$$\dot{x}=ax-bx^3+\sqrt{2\tilde{D}(x,\tau)}\xi(t)\ , \tag{6.9}$$

where

$$\tilde{D}(x,\tau)=D(1+Rx^2+2\tau bx^2)/(1+2\tau bx^2)\ . \tag{6.10}$$

The choice in Eq. (6.10) yields the correct limiting diffusive behavior in Eq. (6.8). Moreover, with the generalized potential being $\Phi(x,\tau)=\Phi_L(x,\tau)$, the limiting behaviors for $\tau\to\infty$ and $\tau=0$ agree as well.

C. Escape times

The crossover theory in Eq. (6.9) incorporates both the correct large-τ behavior in Sec. V A and the correct behavior in Sec. V B for $\tau=0$. With the effective one-dimensional Markov approximation in hand, the escape time follows via Eq. (2.4) in terms of two quadratures. The results of the crossover theory in (6.9), together with the limiting approximations derived in Eq. (5.5) (large τ), and in Eq. (5.12) (small τ), are depicted in Fig. 6. We find that the exact limiting behaviors for $a\tau\ll 1$, and $\tau\gg 1$ are correctly reproduced by the crossover theory and by the corresponding colored noise approximation schemes. In agreement with theory, the escape time is maximal for $\tau\to\infty$, approaching the Smoluchowski limit, and monotonically decreases with decreasing noise color τ towards the corresponding limit given by $T(R,\tau=0)=T(R)$. Put differently, for all fixed R values the escape time $T(R,\tau)$ increases monotonically with increasing noise color to the maximal Smoluchowski result in Eq. (1.3). The behavior at small noise color has been discussed already from a general viewpoint in Sec. V: With weak noise D, the behavior is controlled by the Arrhenius energy which increases proportionally to τ^2. Because our approximation schemes are guaranteed to be accurate to leading exponential order only, the dependence on noise color for prefactors cannot be trusted.

Of special interest are the limiting (exponential) behaviors for which the colored noise approximation schemes assume most accurate results. For example, the approach towards the limiting large-τ Smoluchowski result is evaluated as

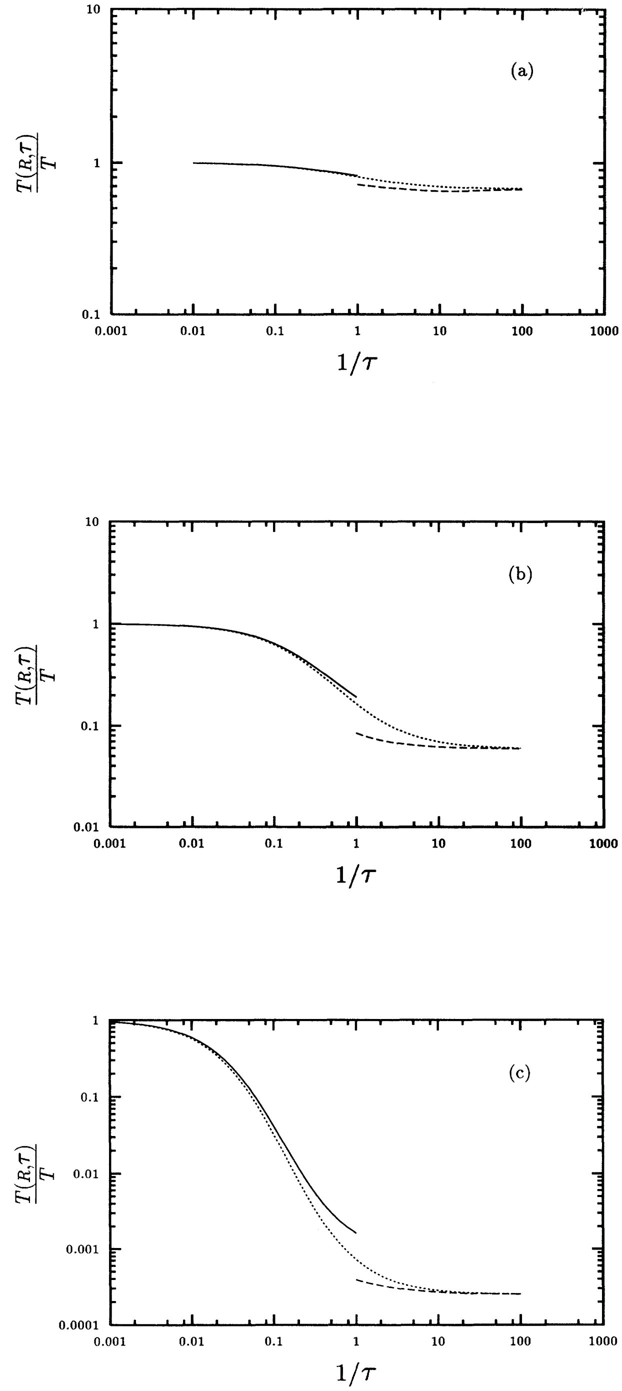

FIG. 6. The ratio of the numerically evaluated [see the quadrature formula in Eq. (2.4)] escape times $T(R,\tau)/T$, with $T=T(R=0)$ being the Smoluchowski value, is depicted versus inverse noise color τ for different noise ratios $R=0.1$ (a), $R=1$ (b), and $R=10$ (c). The results are depicted for the fluctuating double well in Eq. (1.5) with $a=b=1$ for a dimensionless noise strength $D=0.02$; implying for the Smoluchowski escape time the value $T=1.23\ 10^6$ sec. The dotted line depicts the crossover theory in Eq. (6.9), the dashed line equals the small τ theory in Eq. (5.12), and the large noise color approximation in Eq. (5.5) is shown by the solid line.

$$T(R,a\tau \gg 1)=T(R=0)\times\exp\left\{\frac{a^2R}{8Db\tau}\left[-1+\frac{R\ln(a\tau)}{\tau b}+\frac{R}{2\tau b}+O(\tau^{-2})\right]\right\}. \quad (6.11)$$

The behaviors for $D\to 0$, i.e., $R\to\infty$, are calculated to read

$$T(R\to\infty,a\tau<1)=T(R)\exp\left\{\frac{\tau^2}{Q}\left[\frac{a^2b}{3R}+O(R^{-2})\right]\right\}, \quad (6.12)$$

being in accordance with the general behavior in (5.18) For large but finite noise color $a\tau>1$, one finds instead

$$T(R\to\infty,a\tau>1) =T(R)\exp\left\{\frac{\tau}{Q}\left[\frac{a^2}{2}+\frac{ab}{R}[3-a\tau-2\ln(aR/b)]+O(R^{-2},R^{-2}\ln R)\right]\right\}. \quad (6.13)$$

This latter result in Eq. (6.13) has a nice physical explanation. With $R\to\infty$ *and* $a\tau\gg 1$ the fluctuations for x and ζ are small, and the stochastic dynamics follows closely the deterministic flow lines, cf. Fig. 2(c). The escape thus takes place along the curve of turning points $s(x)$ in (6.3), bringing the random walker close to the point $(x=0,\zeta=-a)$. A small noise intensity D then kicks the walker across the separatrix $x=0$, near $\zeta=-a$. With the stationary probability $\rho(\zeta)$ obeying

$$\rho(\zeta)=(\tau/2\pi Q)^{1/2}\exp[-\tau\zeta^2/2Q] ,$$

the escape time to reach $\zeta=-a$, when started out at $(\zeta=0,x=-\sqrt{a/b})$, is thus dominated by the Arrhenius energy $1/\rho(\zeta=-a)$. Thus the escape time

$$T(R\to\infty,a\tau>1)\propto\exp\left[\frac{\tau a^2}{2Q}\right], \quad (6.14)$$

being in agreement with (6.13).

VII. CONCLUSIONS

In this work we have investigated by analytical means bistable stochastic flows that are driven by white and simultaneously by multiplicative, colored noise. The results have been applied to the archetype model of a white noise driven Landau-Ginzburg model (symmetric double well) with a stochastically varying barrier curvature. Our findings have been summarized already in the introduction. One of our prime findings is certainly that use of the refined nonlinear process $\dot{u}(t)$ in Eq. (5.1) leads to different Markovian approximation schemes for the stationary properties of colored noise driven flows. In doing so, we have generalized the commonly used UCNA theory [21,28,29,32] for situations with two noise sources acting.

This Markov approximation yields exact results for parabolic potentials for all values of the noise correlation time τ. The conventional UCNA theory is recovered from our theory if $D\to 0$, i.e., $R\to\infty$. Upon inspecting the smooth crossover results in Fig. 6 we surmise that our approximation also yields reliable estimates for moderate noise color $a\tau\sim O(1)$, i.e., also in regimes away from the exact limiting behaviors as $\tau\to 0$ and $\tau\to\infty$. With the exception of the case with a parabolic potential, there exists presently no effective Markovian approximation yielding the *exact* non-Markovian barrier height governing the exponential leading part of the escape time. Nevertheless we have good confidence in the above conjecture. This conjecture obtains support by noting that a nonzero noise intensity $D>0$ acts in a "stabilizing" way in the sense that the effective friction and/or the effective diffusion in Eqs. (5.5), (6.4), and (6.10) does not attain a singular behavior within regions of local instability. Our study in Fig. 6 for the escape time in a fluctuating double well has motivated others to test the accuracy of our results for moderate noise color values by means of analog simulations [46]; their preliminary experimental results show good agreement with our analytical theory. Notably, there are numerous other contexts for which our results are of relevance. Some are in the limelight of present research activities, such as the description of directed transport in Brownian ratchets [19] and two-noise driven nonlinear optical systems [15,21(b),23].

ACKNOWLEDGMENTS

This work has been supported by the Volkswagen Foundation project "Resonanzaktivierung" (P.H.) (Az. I/65037-1), the DAAD-FAPEsP exchange program (P.H.), and the Brasilian Council of Scientific and Technological Development, CNPq (A.J.R.M.). We also wish to thank Lutz Schimansky-Geier, Peter Reimann, and Roland Bartussek for helpful discussions.

[1] K. Binder and A. P. Young, Rev. Mod. Phys. **58**, 801 (1986).
[2] J. Wong, and C. A. Angell, *Glass: Structure by Spectroscopy* (Dekker, New York, 1976).
[3] H. Frauenfelder and P. G. Wolynes, Science **229**, 337 (1985).
[4] D. L. Stein, Proc. Natl. Acad. Sci. U.S.A. **82**, 3670 (1985).
[5] R. Kubo, Rep. Prog. Phys. **29**, 255 (1966).
[6] A. Einstein, Ann. Phys. (Leipzig) **17**, 549 (1905); N. Niquist, Phys. Rev. **32**, 110 (1928).
[7] P. Hänggi, J. Stat. Phys. **42**, 105 (1986).
[8] P. Hänggi, P. Talkner, and M. Borkovec, Rev. Mod. Phys.

62, 251 (1990).

[9] D. L. Stein, R. G. Palmer, J. L. van Hemmen, and Ch. Doering, Phys. Lett. A **136**, 353 (1989).

[10] D. L. Stein, Ch. Doering, R. G. Palmer, J. L. van Hemmen, and R. McLaughlin, J. Phys. A **23**, L203 (1990).

[11] (a) C. R. Doering and J. C. Gadoua, Phys. Rev. Lett. **69**, 2318 (1992); (b) U. Zürcher and C. R. Doering, Phys. Rev. E **47**, 3862 (1993); (c) C. Van den Broeck, *ibid.*, **47**, 4579 (1993); (d) M. Bier and R. Dean Astumian, Phys. Rev. Lett. **71**, 1649 (1993); (e) P. Reimann, Phys. Rev. E **49**, 4938 (1994); (f) P. Hänggi, Chem. Phys. **180**, 157 (1994); (g) P. Pechukas and P. Hänggi, Phys. Rev. Lett. **73**, 2772 (1994).

[12] P. Hänggi, Phys. Lett. **78A**, 304 (1980).

[13] T. Fonesca, P. Grigolini, and P. Marin, Phys. Lett. **88A**, 117 (1982).

[14] S. Faetti, P. Grigolini, and F. Marchesoni, Z. Phys. B **47**, 353 (1982); H. Fujisaka and S. Groβmann, *ibid.* **43**, 69 (1981); W. Horsthemke and R. Lefever, Phys. Lett. **106A**, 10 (1984); J. M. Sancho, R. Manella, P. V. E. McClintock, and F. Moss, Phys. Rev. A **32**, 3639 (1985).

[15] P. Jung, Th. Leiber, and H. Risken, Z. Phys. B **66**, 397 (1987); **68**, 123 (1987); R. F. Fox, and R. Roy, Phys. Rev. A **35**, 1838 (1987); M. Aguado and M. San Miguel, *ibid.* **37**, 450 (1988).

[16] P. Hänggi, F. Marchesoni, and P. Grigolini, Z. Phys. B **56**, 333 (1984); P. Hänggi and P. S. Riseborough, Phys. Rev. A **27**, 3379 (1983); P. Hänggi, P. Jung, and F. Marchesoni, J. Stat. Phys. **54**, 1367 (1988).

[17] For a review, see Sec. VIII C in Ref. [8], and references given therein.

[18] K. Lindenberg, B. J. West, and G. P. Tsironis, Rev. Solid State Sci. **3**, 143 (1989).

[19] S. Leibler, Nature **370**, 412 (1994), and references therein.

[20] A. Schenzle and H. Brand, Phys. Lett. **69A**, 313 (1979); Phys. Rev. A **20**, 1628 (1979).

[21] (a) P. Jung and P. Hänggi, Phys. Rev. A **35**, 4464 (1987); (b) J. Opt. Soc. Am. **135**, 979 (1988).

[22] K. M. Rattray and A. J. McKane, J. Phys. A **24**, 1215 (1991); cf. ansatz in Eqs. (37) and (38) therein.

[23] E. Peacock-Lopez, F. J. de la Rubia, B. J. West, and K. Lindenberg, Phys. Lett. A **136**, 96 (1989).

[24] J. M. Sancho, M. San Miguel, S. L. Katz, and J. D. Gunton, Phys. Rev. A **26**, 1589 (1982).

[25] P. Hänggi, T. J. Mroczkowski, F. Moss, and P. V. E. McClintock, Phys. Rev. A **32**, 695 (1985).

[26] P. Hänggi, in *Noise in Nonlinear Dynamical Systems,* edited by F. Moss, and P. V. E. McClintock (Cambridge University Press, Cambridge, England, 1989), pp. 307–328.

[27] P. Hänggi, F. Marchesoni, and P. Grigolini, Z. Phys. B **56**, 333 (1984); see Eq. (3.13) and text following therein.

[28] L. H'walisz, P. Jung, P. Hänggi, P. Talkner, and L. Schimansky-Geier, Z. Phys. B **77**, 471 (1989).

[29] R. F. Fox, Phys. Rev. A **37**, 450 (1988); Th. Leiber, F. Marchesoni, and H. Risken, *ibid.* **38**, 983 (1988); L. Schimansky-Geier and Ch. Zülicke, Z. Phys. B **79**, 451 (1990); Li Cao, D. Wu, and X. Luo, Phys. Rev. A **47**, 571 (1993), T. G. Venkatesh and L. M. Patniak, Phys. Rev. E **48**, 2402 (1993).

[30] M. San Miguel and J. M. Sancho, Phys. Lett. **76A**, 97 (1980).

[31] H. Dekker, Phys. Lett. **90A**, 26 (1982); R. Fox, *ibid* **94A**, 281 (1983).

[32] H. S. Wio, P. Colet, M. San Miguel, L. Pesquera, and M. A. Rodriguez, Phys. Rev. A **40**, 7212 (1989); see end of Sec. III therein.

[33] See Eq. (4.57) in Ref. [8].

[34] N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry,* 2nd ed. (North-Holland, Amsterdam, 1992); note the exercise on p. 243 to construct a Fokker-Planck equation for the two-noise driven flow in (1.5).

[35] J. F. Luciani and A. D. Verga, J. Stat. Phys. **50**, 567 (1988).

[36] R. Graham, Z. Phys. B **26**, 281 (1977).

[37] P. Hänggi, Z. Phys. B **75**, 275 (1989); in *Path Integrals from meV to MeV: Tutzing, 1992;* edited by H. Grabert, A. Inomata, L. S. Schulman, and U. Weiss (World Scientific, Singapore, 1993), pp. 289–301.

[38] Note that the extrema of $p(x)$ in the presence of multiplicative noise become shifted within order $O(D)$, due to the nonconstant diffusion.

[39] W. Horsthemke and R. Lefever, *Noise Induced Transitions* (Springer-Verlag, Berlin, 1984); see Secs. 6 and 7.1 therein.

[40] S. Kabashima, S. Kogure, T. Kawakubo, and T. Okada, J. Appl. Phys. **50**, 6296 (1979); S. Kabashima and T. Kawakubo, Phys. Lett. **A70**, 375 (1979).

[41] See Eq. (4.57) in Ref. [8] with $T(R)\equiv k^{-1}$.

[42] R. L. Stratonovitch, *Topics in the Theory of Random Noise* (Gordon & Breach, New York, 1963), Vol. I; see Eq. (4.134) therein.

[43] For a more recent presentation of this widely used approximation, see the various contributions in *Noise in Nonlinear Dynamical Systems,* edited by F. Moss and P. V. E. McClintock (Cambridge University Press, Cambridge, England, 1989), Vol. I.

[44] See Eq. (22) in Ref. [31], where for our case $\varepsilon_{12}=\tau$, $\eta_{12}=\tau$, $\delta_1^1=x\sqrt{Q}$, $\delta_2^1=\sqrt{D}$, $K_{12}^1=-K_{21}^1=\sqrt{QD}$.

[45] P. Reimann, Phys. Rev. E **49**, 4938 (1994); see Eq. (20) therein.

[46] F. Marchesoni, L. Gammaitoni, E. Menichella-Saetta, and S. Santucci (unpublished).